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Hypermultiplets and hypercomplex geometry from 6 to 3 dimensions

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Abstract

The formulation of hypermultiplets that has been developed for 5-dimensional matter multiplets is by dimensional reductions translated into the appropriate spinor language for 6 and 4 dimensions. We also treat the theories without actions that have the geometrical structure of hypercomplex geometry. The latter is the generalization of hyper-Kähler geometry that does not require a Hermitian metric and hence corresponds to field equations without action. The translation tables of this paper allow the direct application of superconformal tensor calculus for the hypermultiplets using the available Weyl multiplets in 6 and 4 dimensions. Furthermore, the hypermultiplets in 3 dimensions that result from reduction of vector multiplets in 4 dimensions are considered, leading to a superconformal formulation of the **c**-map and an expression for the main geometric quantities of the hyper-Kähler manifolds in the image of this map.

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Contents

1 Introduction

It has been known for a long time that hypermultiplets in theories with 8 fermionic supercharges are related to quaternionic geometry [1], which has found various applications. In the context of the construction of 5-dimensional supergravity-matter couplings a new formulation for hypermultiplets has been given [2]. The starting point for this construction is the realization of the supersymmetry algebra, rather than the invariance of a proposed action. This new point of view leads to a more manifest geometrical formulation of the hypermultiplets. Thereby the ingredients of a hypercomplex structure are obtained first, before a metric appears. The metric is introduced afterwards, when in a second stage the action is considered, leading to hyper-Kähler manifolds. On the physics side, this treatment has the advantage that theories whose field equations are not necessarily derivable from an action principle are included in this analysis. Moreover, there is a transparent way in which supergravity couplings are obtained. A hypercomplex or hyper-Kähler manifold can lead to a supergravity theory if it allows a conformal structure, mathematically expressed as the presence of a closed homothetic Killing vector. Such a vector allows a straightforward use of the superconformal tensor calculus, leading to a supergravity theory as has been illustrated in [3].

It turns out that the supergravity action and corresponding geometry are completely fixed once the rigid one is determined, and this conformal structure (closed homothetic Killing vector) is known. The supergravity theory is obtained using superconformal tensor calculus (see for instance [4–7]). This construction goes by coupling the hypermultiplets (or other matter multiplets) to a multiplet containing the gauge fields of the superconformal symmetries, called the 'Weyl multiplet'. After gauge fixing conformal symmetries, one obtains the couplings of the matter multiplet to Poincaré supergravity. If multiplets with gauge fields are included, then also the couplings with gauged isometries are obtained using the same steps. Especially for hypermultiplets, the latter procedure will be clarified in [8].

It has been mentioned in [2] that this development is independent of the application to 5 dimensions, and could also be applied to 6 and 4 dimensions. However, this generalization is not straightforward due to the fact that spinors are described in 6 dimensions as symplectic-Weyl spinors, in 5 dimensions as symplectic spinors and in 4 dimensions as Majorana spinors. Therefore the appearance of the geometric quantities differs (see for instance [9], where an analysis of rigid hypermultiplets with action is performed in 4 dimensions, giving rise to hyper-Kähler geometry). In order to be able to use the general analysis of [2] in other dimensions, one needs a translation table. Especially for applications of supersymmetry and supergravity in 4 dimensions such a translation is of practical use, and on the other hand not straightforward. In this paper we will generalize the results from [2] to 4 and 6 dimensions through dimensional reduction of the transformation rules of the 5-dimensional theory. This will lead to connections between the different geometrical quantities in the theories in 4, 5 and 6 dimensions. The advances made in 5-dimensional supersymmetry and supergravity are then immediately applicable for 4 and 6 dimensions. Indeed, the superconformal tensor calculus for 4 dimensions [10–13] and 6 dimensions [14], with especially the formulation of the Weyl and vector multiplets that we need for this programme, has already been known for a long time. Therefore, the formulation of the basic elements of rigid hypermultiplets in this language is all that is needed.

In the second part of this paper we will start from the four-dimensional vector multiplet, whose geometry (known as special geometry) is fixed by a holomorphic function of the complex scalars in the vector multiplet. Reducing it to 3 dimensions gives again a hypermultiplet. The dimensional reduction from four to three dimensions in supergravity gives rise to the **c**-map between special

Kähler geometry and quaternionic-Kähler geometry. This map was first introduced in [15]. While it is obtained in supergravity by dimensional reduction from 4 to 3 dimensions, it is obtained in string theory by a T-duality between type IIA and type IIB strings before dimensionally reducing both string theories from 10 to 4 dimensions. This procedure exchanges vector multiplets with hypermultiplets and thus provides a mapping between special Kähler and quaternionic-Kähler geometry. This **c**-map leads to the notion of '*special quaternionic-Kähler manifolds*', which are those manifolds appearing in the image of the **c**-map. They are a subclass of the quaternionic-Kähler manifolds. In rigid supersymmetry a similar map has been considered in [16] by dimensional reduction from 4 to 3 dimensions. The resulting subclass of hyper-Kähler manifolds in the image of this map has been called 'special hyper-Kähler' manifolds[1]. We will reconsider this map using the geometric formulation as first developed in 5 dimensions and using the same generalization to other dimensions as in the first part of the paper.

In section 2 we will start from the 5-dimensional hypermultiplet. After reducing it to 4 dimensions we obtain translations of the geometric quantities defining the hypercomplex geometry. Then we will do the same analysis again for the 6-dimensional hypermultiplet. Equations of motion are derived through closure of the supersymmetry algebra.

Section 3 starts with a summary of the dimensional reduction of the rigid supersymmetry transformations of the 4-dimensional vector multiplet to 3 dimensions [16], leading to the rigid **c**-map discussed above. We will use the obtained translation formulae to lift the resulting hypermultiplet up to 5 dimensions. This facilitates the identification of the main geometrical quantities, due to the more transparent notation in 5 dimensions. We obtain in this way general formulae for the curvature tensors of special hyper-Kähler manifolds.

2 Hypermultiplets in $d = 5, 4, 6$

In this section, we review hypermultiplets in $4, 5$ and 6 dimensions, without assuming the existence of an action. We start by considering the transformations of the hypermultiplet in $d = 5$ and 4 under rigid supersymmetry. Then we reduce the five-dimensional theory to 4 dimensions. This allows us to relate the geometrical quantities that arise in both dimensions. Next, we lift the five-dimensional theory up to $d = 6$. In this way, we are able to show that each time, the scalars describe a hypercomplex manifold, which appears in different guises, depending on the dimension considered. We also derive equations of motion for the fermions through closure of the supersymmetry algebra.

2.1 Hypermultiplets in $d = 5$

Here we review the description of hypermultiplets in five dimensions. An interesting discussion of this can be found in [2]. For calculations with spinors we will mainly use the conventions of [17].

A system of r hypermultiplets consists of $4r$ real scalars $q^X(x)$, $X = 1, \cdots, 4r$ and $2r$ spinors $\zeta^A(x)$, $A = 1, \cdots, 2r$. In five dimensions, the spinors are subject to symplectic Majorana reality conditions. One introduces matrices $\rho_A{}^B$ and $E_i{}^j$, obeying

$$\rho\rho^* = -\mathbb{1}_{2r}, \qquad EE^* = -\mathbb{1}_2. \tag{2.1}$$

[1]Observe that the **c**-map for rigid Kähler geometry does not lead directly to the **c**-map for local Kähler geometry. The latter increases the dimension from n complex to $n + 1$ quaternionic. The extra quaternion originates from the dimensional reduction of pure supergravity in 4 dimensions. The map that we consider here does therefore *not* lead to a hyper-Kähler manifold that can be mapped to a quaternionic-Kähler manifold by using superconformal tensor calculus.

Using these, symplectic Majorana conditions for the fermions and supersymmetry parameters can be defined as

$$(\zeta^A)^* = \alpha\mathcal{C}_5\gamma_0\zeta^B\rho_B{}^A, \qquad (\epsilon^i)^* = \alpha\mathcal{C}_5\gamma_0\epsilon^j E_j{}^i, \tag{2.2}$$

where $\mathcal{C}_5$ is the charge conjugation matrix in 5 dimensions and α is an irrelevant number of modulus 1. We will always use the basis where $E_i{}^j = \varepsilon_{ij}$. Indices are raised and lowered using the north-west-south-east convention.

The supersymmetry transformations take the general form

$$\begin{aligned}
\delta(\epsilon)q^X &= -\mathrm{i}\bar{\epsilon}^i\zeta^A f^X_{iA}, \\
\delta(\epsilon)\zeta^A &= \frac{1}{2}\mathrm{i}\gamma^\mu\partial_\mu q^X f^{iA}_X\epsilon_i - \zeta^B\omega_{XB}{}^A(\delta(\epsilon)q^X).
\end{aligned} \tag{2.3}$$

Here, $f^X_{iA}(q)$, $f^{iA}_X(q)$ and $\omega_{XB}{}^A(q)$ are arbitrary functions of the scalars. Due to reality of the scalars and the symplectic Majorana conditions for the fermions, they satisfy the following reality conditions

$$(f^{iA}_X)^* = f^{jB}_X E_j{}^i\rho_B{}^A, \qquad (\omega_{XA}{}^B)^* = (\rho^{-1}\omega_X\rho)_A{}^B. \tag{2.4}$$

In order that the commutator of 2 supersymmetry transformations gives a translation, one has to impose

$$\begin{aligned}
f^{iA}_Y f^X_{iA} &= \delta^X_Y, \qquad f^{iA}_X f^X_{jB} = \delta^i_j\delta^A_B, \\
\mathcal{D}_Y f^X_{iB} &\equiv \partial_Y f^X_{iB} - \omega_{YB}{}^A f^X_{iA} + \Gamma_{ZY}{}^X f^Z_{iB} = 0,
\end{aligned} \tag{2.5}$$

where Γ is an object symmetric in its lower indices. Note that f^X_{iA} and f^{iA}_X are each others inverse and that they are covariantly constant with connections Γ and ω. These are the only conditions on the target space that follow from imposing closure of the supersymmetry transformations. There are no further geometrical constraints from the commutator of 2 supersymmetries on the fermions. In this respect, this commutator will rather define equations of motion for the hypermultiplet.

Geometry. The geometry of the target space is a hypercomplex manifold. In fact it is a weakened form of hyper-Kähler geometry, where no Hermitian covariantly constant metric is defined. The basic object for defining these manifolds is a triplet of complex structures, the hypercomplex structure

$$\vec{J}_X{}^Y \equiv -\mathrm{i}f^{iA}_X\vec{\sigma}_i{}^j f^Y_{jA}. \tag{2.6}$$

These structures are covariantly constant and satisfy the quaternion algebra. The statement for arbitrary 3-vectors $\vec{\alpha}$ and $\vec{\beta}$,

$$\vec{\alpha}\cdot\vec{J}\,\vec{\beta}\cdot\vec{J} = -\mathbb{1}_{4r}\vec{\alpha}\cdot\vec{\beta} + \left(\vec{\alpha}\times\vec{\beta}\right)\cdot\vec{J}. \tag{2.7}$$

We adopt the following definitions for the curvature tensors of the Γ and ω connections:

$$R_{XYZ}{}^W \equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V, \tag{2.8}$$

$$\mathcal{R}_{XYB}{}^A \equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C. \tag{2.9}$$

The integrability conditions on the vielbeins f^{iA}_X relate the curvature tensors $R_{XYZ}{}^W$ and $\mathcal{R}_{XYB}{}^A$:

$$R_{XYZ}{}^W = f^W_{iA}f^{iB}_Z\mathcal{R}_{XYB}{}^A, \qquad \delta^i_j\mathcal{R}_{XYB}{}^A = f^{iA}_W f^Z_{jB}R_{XYZ}{}^W. \tag{2.10}$$

Using cyclicity properties of the Riemann tensor, one can also obtain

$$\begin{aligned} f_{Ci}^{X} f_{jD}^{Y} \mathcal{R}_{XYB}{}^{A} &= -\tfrac{1}{2}\varepsilon_{ij} W_{CDB}{}^{A}, \\ W_{CDB}{}^{A} &\equiv f_{C}^{iX} f_{iD}^{Y} \mathcal{R}_{XYB}{}^{A} = \tfrac{1}{2} f_{C}^{iX} f_{iD}^{Y} f_{jB}^{Z} f_{W}^{Aj} R_{XYZ}{}^{W}. \end{aligned} \tag{2.11}$$

This tensor W is symmetric in its 3 lower indices.

Dynamics. After calculating the commutator of 2 supersymmetries on the fermions, one notes that the algebra does not close. Demanding that non-closure terms vanish leads to equations of motion for the fermions:

$$\Gamma^{A} = \gamma^{a}\mathcal{D}_{a}\zeta^{A} + \tfrac{1}{2} W_{CDB}{}^{A} \zeta^{B} \bar{\zeta}^{D} \zeta^{C} = 0, \tag{2.12}$$

where the covariant derivative is given by

$$\mathcal{D}_{\mu}\zeta^{A} \equiv \partial_{\mu}\zeta^{A} + (\partial_{\mu} q^{X})\zeta^{B}\omega_{XB}{}^{A}. \tag{2.13}$$

So one sees that imposing the supersymmetry algebra on the scalars leads to defining the hypercomplex geometry, while closure on the fermions leads to equations of motion for these fermions.

Projection to supergravity. A closed homothetic Killing vector is a vector k^X satisfying

$$\mathcal{D}_{Y} k^{X} = \frac{d-2}{2}\delta_{Y}{}^{X} = \frac{3}{2}\delta_{Y}{}^{X}. \tag{2.14}$$

This induces a conformal symmetry, where e.g. the dilatations with parameter λ_D act on the hypermultiplet scalar as

$$\delta_{D}(\lambda_{D}) q^{X} = \lambda_{D}\left(x^{\mu}\partial_{\mu} q^{X} + k^{X}\right). \tag{2.15}$$

The vector k^X and the 3 vectors $k^Y \vec{J}_Y{}^X$ define 4 scalars that form a quaternion. After gauge-fixing the superconformal symmetry using the Weyl multiplet, this quaternion is gauge-fixed, together with its fermionic partner, such that a supergravity theory coupled to hypermultiplets remains. Clearly the hypermultiplet sector has one less quaternionic dimension in the supergravity theory than in the corresponding rigid supersymmetry theory that was its starting point. This is analysed in detail in [8].

2.2 Hypermultiplets in $d = 4$

Superconformal hypermultiplets in four dimensions were discussed in [9]. We will review the supersymmetry transformations and the resulting hypercomplex geometry and equations of motion.

The main difference with the 5-dimensional case lies in the reality conditions that one imposes on the fermions. Whereas in five dimensions one is obliged to impose symplectic Majorana conditions, in four dimensions one has Majorana spinors. Moreover, one can write the Majorana spinors as chiral spinors. So a system of r hypermultiplets will now consist of $4r$ real scalars q^X, and $2r$ spinors $\zeta^{\bar{\alpha}}$ with positive chirality and $2r$ spinors ζ^{α} with negative chirality. By complex conjugation, the indices α and $\bar{\alpha}$ are interchanged, so we have indeed a system of $2r$ Majorana spinors. The chirality of the 4-dimensional supersymmetry parameters is indicated by the position of the SU(2)-index

$$\tilde{\epsilon}^{i} = \gamma_5 \tilde{\epsilon}^{i}, \qquad \tilde{\epsilon}_{i} = -\gamma_5 \tilde{\epsilon}_{i}. \tag{2.16}$$

The general form of the supersymmetry transformations is then

$$\begin{aligned}
\delta(\tilde\epsilon)q^X &= \gamma^X_{i\bar\alpha}\tilde\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Xi}_{\alpha}\tilde\epsilon_i\zeta^{\alpha},\\
\delta(\tilde\epsilon)\zeta^{\alpha} &= V^{\alpha}_{Xi}\gamma^{\mu}\partial_{\mu}q^X\tilde\epsilon^i - \delta(\tilde\epsilon)q^X\Gamma_X{}^{\alpha}{}_{\beta}\zeta^{\beta},\\
\delta(\tilde\epsilon)\zeta^{\bar\alpha} &= \bar V^{i\bar\alpha}_{X}\gamma^{\mu}\partial_{\mu}q^X\tilde\epsilon_i - \delta(\tilde\epsilon)q^X\bar\Gamma_X{}^{\bar\alpha}{}_{\bar\beta}\zeta^{\bar\beta}.
\end{aligned} \tag{2.17}$$

Again the coefficients $\gamma^X_{i\bar\alpha}$, V^{α}_{Xi} and $\Gamma_X{}^{\alpha}{}_{\beta}$ are functions of the scalars q^X. Their complex conjugates are

$$\bar\gamma^{Xi}_{\alpha} = \left(\gamma^X_{i\bar\alpha}\right)^*, \qquad \bar V^{i\bar\alpha}_X = (V^{\alpha}_{Xi})^*, \qquad \bar\Gamma_X{}^{\bar\alpha}{}_{\bar\beta} = (\Gamma_X{}^{\alpha}{}_{\beta})^*. \tag{2.18}$$

Note that complex conjugation raises or lowers here the indices i, j, while indices α are replaced by $\bar\alpha$. Demanding that the commutator of 2 supersymmetries on the scalars gives a translation, leads to the restrictions

$$\begin{aligned}
&\gamma^X_{i\bar\alpha}\bar V^{j\bar\alpha}_Y + \bar\gamma^{Xj}_{\alpha}V^{\alpha}_{Yi} = \delta^j_i\delta^X_Y,\\
&\bar V^{i\bar\alpha}_X\gamma^X_{j\bar\beta} = \delta^i_j\delta^{\bar\alpha}_{\bar\beta}.
\end{aligned} \tag{2.19}$$

The first condition is also known as the Clifford condition. The second condition expresses the invertibility of the vielbeins. Demanding closure of the superalgebra leads to the following relations:

$$\begin{aligned}
\partial_Y\gamma^X_{i\bar\alpha} + \Gamma_{YZ}{}^X\gamma^Z_{i\bar\alpha} - \gamma^X_{i\bar\beta}\Gamma_Y{}^{\bar\beta}{}_{\bar\alpha} &= 0,\\
\partial_Y\bar\gamma^{Xi}_{\alpha} + \Gamma_{YZ}{}^X\bar\gamma^{Zi}_{\alpha} - \bar\gamma^{Xi}_{\beta}\Gamma_Y{}^{\beta}{}_{\alpha} &= 0.
\end{aligned} \tag{2.20}$$

Again the $\Gamma_{XY}{}^Z$ are symmetric in the lower indices (XY). So we see that the vielbeins γ (or $\bar\gamma$) are covariantly constant with respect to connections $\Gamma_X{}^{\alpha}{}_{\beta}$ (or $\bar\Gamma_X{}^{\bar\alpha}{}_{\bar\beta}$) and $\Gamma_{XY}{}^Z$. Again these are the only conditions implied by supersymmetry on the target space. They imply that the manifold is hypercomplex. Closure of the supersymmetry algebra on the fermions leads to equations of motion.

One can define a matrix ρ:

$$\rho^{\alpha}{}_{\bar\beta} \equiv \tfrac{1}{2}\varepsilon^{ij}V^{\alpha}_{Xi}\gamma^X_{j\bar\beta}, \tag{2.21}$$

that is subject to the following conditions:

$$\rho^{\alpha}{}_{\bar\beta}\rho^{\bar\beta}{}_{\gamma} = -\delta^{\alpha}_{\gamma}, \qquad \bar\gamma^{Xi}_{\alpha}\rho^{\alpha}{}_{\bar\beta}\varepsilon_{ij} = \gamma^X_{j\bar\beta}, \qquad \varepsilon_{ij}\rho^{\alpha}{}_{\bar\beta}\bar V^{j\bar\beta}_X = V^{\alpha}_{Xi}. \tag{2.22}$$

The last two equations are reality conditions for the γ and V coefficients. In this respect, this matrix ρ is similar to that used in (2.2), and satisfies the same relation (2.1). However, in 4 dimensions the complex conjugation of the vielbein coefficients is defined in (2.18), and ρ is defined by (2.21) while in 5 dimensions the vielbeins and their complex conjugates are in the same tensors f^{iA}_X related by ρ as in (2.4).[2]

The discussion in [9] implied the existence of an action and thus a metric for the target space. The resulting geometry was that of a hyper-Kähler manifold. Here we will no longer demand the existence of an action. We expect to find again the hypercomplex manifold discussed in the previous section. Again, we expect to find a triplet of complex structures $\vec J$ that satisfies the quaternion

[2] This difference is relevant when comparing the calculation of commutators on scalars between 4 and 5 dimensions. Note that the result (2.5) is already obtained from the commutator on the scalars in 5 dimensions, while in 4 dimensions we need the commutators on scalars and fermions to arrive at (2.19). The difference is that the reality relation (2.4) follows from the properties of symplectic Majorana spinors in 5 dimensions. The analogous relation in 4 dimensions is only derived after the further steps explained above.

algebra (2.7). One can also calculate the necessary curvature tensors. The integrability condition on the vielbeins now leads to

$$\begin{aligned} R_{XYV}{}^{Z}\bar{\gamma}_{\beta}^{Vi}V_{jZ}^{\gamma} &= \delta_j^i \mathcal{R}_{XY}{}^{\gamma}{}_{\beta}, \\ R_{XYV}{}^{Z}\gamma_{i\bar{\beta}}^{V}\bar{V}_Z^{j\bar{\gamma}} &= \delta_i^j \mathcal{R}_{XY}{}^{\bar{\gamma}}{}_{\bar{\beta}}, \end{aligned} \tag{2.23}$$

where the curvature tensors are defined in a way similar to that in five dimensions.

The curvature tensors are dependent on a conformal tensor W, which we also introduced in the 5-dimensional case. It determines both $\mathcal{R}_{XY}{}^{\alpha}{}_{\beta}$ and $R_{XYZ}{}^{W}$. It can be defined by

$$W_{\gamma\delta}{}^{\alpha}{}_{\beta} \equiv \tfrac{1}{4}\varepsilon_{ij}\bar{\gamma}_{\gamma}^{Xj}\bar{\gamma}_{\delta}^{Yi}\mathcal{R}_{XY}{}^{\alpha}{}_{\beta} = \tfrac{1}{8}\varepsilon_{ij}\bar{\gamma}_{\gamma}^{Xj}\bar{\gamma}_{\delta}^{Yi}\bar{\gamma}_{\beta}^{Vk}V_{kZ}^{\alpha}R_{XYV}{}^{Z}. \tag{2.24}$$

(We take normalizations such that it is the dimensional reduction of the definition that we have in 5 dimensions, see below.) The tensor is symmetric in the 3 lower indices, as follows from cyclicity properties of the Riemann tensor. One can also define this tensor with $\bar{\alpha}$-indices instead of α-indices by multiplication with $\rho^{\kappa}{}_{\bar{\alpha}}$. For instance

$$W_{\bar{\alpha}\beta}{}^{\gamma}{}_{\delta} = W_{\kappa\beta}{}^{\gamma}{}_{\delta}\rho^{\kappa}{}_{\bar{\alpha}} = -\tfrac{1}{4}\gamma_{i\bar{\alpha}}^{X}\bar{\gamma}_{\beta}^{Yi}\mathcal{R}_{XY}{}^{\gamma}{}_{\delta}. \tag{2.25}$$

As in the five-dimensional case, calculating the commutator of 2 supersymmetry transformations on the fermions leads to non-closure terms, corresponding to dynamical equations for these fermions. One obtains

$$\gamma^{\mu}\mathcal{D}_{\mu}\zeta^{\alpha} + \tfrac{1}{2}W_{\bar{\gamma}\delta}{}^{\alpha}{}_{\beta}\zeta^{\bar{\gamma}}\bar{\zeta}^{\delta}\zeta^{\beta} = 0. \tag{2.26}$$

The covariant derivative is given by

$$\mathcal{D}_{\mu}\zeta^{\alpha} = \partial_{\mu}\zeta^{\alpha} + (\partial_{\mu}q^{X})\Gamma_{X}{}^{\alpha}{}_{\beta}\zeta^{\beta}. \tag{2.27}$$

2.3 Dimensional reduction from 5 to 4 dimensions

In this section, we perform the process of dimensional reduction in order to obtain the 4-dimensional theory from the 5-dimensional one. We first give a brief sketch of how this procedure works. Then we apply this to obtain relations between the geometrical quantities appearing in both dimensions. Finally, we show how these relations lead to translations of the constraints defining the hypercomplex geometry in the different dimensions. This will allow us to conclude that indeed the same geometrical relations appear, but in different guises.

2.3.1 Some notes on the reduction

In order to perform the reduction, we will suppose that the fields are independent of the fifth spacetime coordinate. We also have a set of five γ-matrices. In four dimensions, we will use the first four of these matrices to form the four-dimensional Clifford algebra. The fifth γ-matrix will be used to form projection operators $P_L = \frac{1}{2}(1+\gamma_5)$ and $P_R = \frac{1}{2}(1-\gamma_5)$, allowing us to split 4-dimensional spinors into chiral spinors. In 5 dimensions the spinors $\zeta^A(x)$ are subject to symplectic Majorana conditions, while in 4 dimensions we assume Majorana reality conditions for the fermion fields. The reduction can be obtained by taking the following identification between the 5- and 4-dimensional spinors:

$$\begin{aligned} \sqrt{2}\zeta^{\bar{\alpha}} &= \tfrac{1}{2}(1+\gamma_5)\zeta^{A}, \\ \sqrt{2}\zeta^{\alpha} &= \tfrac{1}{2}(1-\gamma_5)\zeta^{B}\rho_{B}{}^{A}. \end{aligned} \tag{2.28}$$

Note that we use here a special notation where the value of A is the same as that of α or in other cases of $\bar{\alpha}$ (complex conjugate). We still keep the different index notation, as that will make it clear whether the object is a quantity that appears in 5 dimensions or in 4 dimensions. Note that the four-dimensional charge conjugation matrix is given by $\mathcal{C}_4 = \mathcal{C}_5\gamma_5$. A similar identification as in (2.28) applies to the supersymmetry parameters

$$\begin{aligned}\sqrt{2}\tilde{\epsilon}^i &= \tfrac{1}{2}(1+\gamma_5)\epsilon^i, \\ \sqrt{2}\tilde{\epsilon}_i &= \tfrac{1}{2}(1-\gamma_5)\epsilon^j E_j{}^i.\end{aligned} \tag{2.29}$$

2.3.2 Connections between 5- and 4-dimensional quantities

The scalars q^X of the 4 and 5-dimensional hypermultiplets are trivially identified, and reducing the supersymmetry transformation laws according to the above rules, we can relate the geometrical quantities appearing in the 4 and 5-dimensional theories. We obtain the following translation formulae between the vielbeins appearing in the supersymmetry transformation rules in the different dimensions considered:

$$f_{iA}^X = \frac{\mathrm{i}}{2}\gamma_{i\bar{\alpha}}^X = \frac{\mathrm{i}}{2}\rho^\beta{}_{\bar{\alpha}}\bar{\gamma}_\beta^{Xj}\varepsilon_{ji}. \tag{2.30}$$

For the inverse vielbeins we get

$$f_X^{iA} = -2\mathrm{i}\bar{V}_X^{i\bar{\alpha}} = 2\mathrm{i}\varepsilon^{ki}\rho^{\bar{\alpha}}{}_\beta V_{Xk}^\beta. \tag{2.31}$$

We can also relate the spin connections in the 4 and 5-dimensional formalism

$$\bar{\Gamma}_X{}^{\bar{\alpha}}{}_{\bar{\beta}} = \omega_{XB}{}^A, \qquad \Gamma_X{}^\delta{}_\gamma = (\rho^{-1}\omega_X\rho)_C{}^D. \tag{2.32}$$

The reality conditions of the different quantities of the five-dimensional theory are consistent with those of the quantities of the 4-dimensional formalism. Thus, we have obtained the necessary relations between the fundamental quantities defining the hypermultiplet.

2.3.3 Reduction of geometrical quantities

In this section, we will apply the previously derived formulae to relate the geometrical constraints and quantities in the different languages. By using the identifications (2.30) and (2.31) in the equation $f_X^{iA}f_{jB}^X = \delta_j^i\delta_B^A$, one obtains the second condition in (2.19). By using the same vielbein identifications and the first relation of (2.5) one can get the Clifford condition of the four-dimensional theory. Indeed, first for $i=j$, using the obtained identifications, we get

$$\gamma_{i\bar{\alpha}}^X\bar{V}_Y^{i\bar{\alpha}} + \bar{\gamma}_\alpha^{Xi}V_{Yi}^\alpha = f_{iA}^X f_Y^{iA} + f_{lB}^X f_Y^{lB} = 2\delta_Y^X. \tag{2.33}$$

Considering the same condition with $i \neq j$ gives

$$\gamma_{i\bar{\alpha}}^X\bar{V}_Y^{j\bar{\alpha}} + \bar{\gamma}_\alpha^{Xj}V_{Yi}^\alpha = f_Y^{jA}f_{iA}^X - f_Y^{jB}f_{iB}^X, \tag{2.34}$$

as expected. It is also easy to see that the covariant constancy of the vielbeins in the four-dimensional formalism is just the translation of the covariant constancy of the vielbeins in the five-dimensional theory.

From the identifications (2.32) for the spin connections, it follows directly that the definition of $\mathcal{R}_{XY}{}^{\bar{\alpha}}{}_{\bar{\beta}}$ is the translation of the definition of $\mathcal{R}_{XYB}{}^A$. Using the vielbein identifications, one

also obtains (2.23) by translating (2.10). One can also show that the curvature tensor $\mathcal{R}_{XY}{}^{\alpha}{}_{\beta}$ corresponds to the spin curvature of the 5-dimensional formalism in the following way:

$$\mathcal{R}_{XY}{}^{\beta}{}_{\alpha} = (\rho^{-1}\mathcal{R}_{XY}\rho)_A{}^B. \tag{2.35}$$

Note that in five dimensions

$$(\mathcal{R}_{XYA}{}^B)^* = (\rho^{-1}\mathcal{R}_{XY}\rho)_A{}^B. \tag{2.36}$$

Hence, we have

$$\mathcal{R}_{XYA}{}^B = \mathcal{R}_{XY}{}^{\bar{\beta}}{}_{\bar{\alpha}}. \tag{2.37}$$

As mentioned already, the matrix ρ defined in (2.21) is related to the ρ-matrix, used in five dimensions to define the symplectic Majorana conditions. It turns out that

$$\rho_B{}^A = \rho^{\alpha}{}_{\bar{\beta}}, \qquad (\rho_B{}^A)^* = \rho^{\bar{\alpha}}{}_{\beta}. \tag{2.38}$$

In this way, one can obtain the conditions (2.22) by translating the reality conditions for the vielbeins in 5 dimensions (2.4).

As mentioned, the scalars q^X of the 4- and 5-dimensional hypermultiplets are identified. Bosonic quantities as the complex structures can therefore be straightforwardly identified too. The expressions in terms of frame-dependent quantities as in (2.6) are, however, different. The translation of the latter into the 4-dimensional formalism gives

$$\vec{J}_X{}^Y = -\mathrm{i}\bar{V}_X^{j\bar{\alpha}}\vec{\sigma}_j{}^i\gamma^Y_{i\bar{\alpha}}. \tag{2.39}$$

The conformal tensor W as defined in (2.24) is the translation of the five-dimensional W-tensor

$$W_{ABC}{}^D = W_{\bar{\alpha}\bar{\beta}}{}^{\bar{\delta}}{}_{\bar{\gamma}}. \tag{2.40}$$

The translation of the holomorphic W-tensor (with α-indices) can be obtained by multiplication with $\rho^{\kappa}{}_{\bar{\alpha}}$, see (2.25).

It is useful to see that this translation suggests a simplification in the formulation of hypermultiplets in 4 dimensions. Indeed, it turns out that we can restrict ourselves to the quantities with only $\bar{\alpha}$ indices, and no α-indices to describe the full geometry. This $\bar{\alpha}$ index is identified with the A index of the formulation in 5 dimensions. The basic relation is only the second line of (2.19). The objects with α indices are defined as the complex conjugates, see (2.18), and satisfy as in 5 dimensions the reality constraint (2.22). This is e.g. sufficient to derive the first line of (2.19). The objects with α-indices are useful for writing fermions with negative chirality, but as the latter are related to the positive-chirality components by the Majorana condition, they are not independent.

2.4 Hypermultiplets in 6 dimensions

In this section, we will obtain a theory of hypermultiplets in $d = 6$, starting from the 5-dimensional theory. Again, special attention goes to the geometry that arises in these theories. We start by giving some comments about the dimensional reduction. Next, we argue that the constraints implied by supersymmetry on the target space are the same as in 5 dimensions. Finally, equations of motion are derived, without assuming the existence of an action.

2.4.1 Comments on uplifting from 5 to 6 dimensions

In 5 dimensions, a Clifford algebra of γ-matrices can be represented by 4×4-matrices. In six dimensions, one now has 6 Γ-matrices, which are 8×8-matrices. By using the 5-dimensional γ-matrices, one can construct the following representation of the 6-dimensional Clifford algebra.

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_1 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} \gamma_2 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \\ \Gamma_3 &= \begin{pmatrix} \gamma_3 & 0 \\ 0 & \gamma_3 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \qquad \Gamma_6 = \begin{pmatrix} 0 & -\mathrm{i}\gamma_5 \\ \mathrm{i}\gamma_5 & 0 \end{pmatrix}. \end{aligned} \tag{2.41}$$

One can also define a matrix Γ_*:

$$\Gamma_* = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5\Gamma_6 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}. \tag{2.42}$$

This matrix anticommutes with the other Γ-matrices.
One can also define a charge conjugation matrix $\mathcal{C}_6$ in terms of $\mathcal{C}_5$:

$$\mathcal{C}_6 = \begin{pmatrix} 0 & \mathcal{C}_5 \\ \mathcal{C}_5 & 0 \end{pmatrix}. \tag{2.43}$$

In six dimensions, spinors also obey symplectic Majorana reality conditions. Therefore, identifications between the 5- and 6-dimensional spinors are simpler than in the previous case[3]. We take a basis in which γ_5 is diagonal [$\gamma_5 = \mathrm{diag}(1,1,-1,-1)$]. The spinors are mapped as follows:

$$\begin{aligned} \zeta^A = \begin{pmatrix} \zeta_1^A \\ \zeta_2^A \\ \zeta_3^A \\ \zeta_4^A \end{pmatrix} \quad &\rightarrow \quad \tilde{\zeta}^A = \frac{1}{2}(1+\Gamma_*)\tilde{\zeta}^A = \begin{pmatrix} \zeta_1^A \\ \zeta_2^A \\ 0 \\ 0 \\ 0 \\ 0 \\ \zeta_3^A \\ \zeta_4^A \end{pmatrix}, \\ \epsilon^i = \begin{pmatrix} \epsilon_1^i \\ \epsilon_2^i \\ \epsilon_3^i \\ \epsilon_4^i \end{pmatrix} \quad &\rightarrow \quad \tilde{\epsilon}^i = \frac{1}{2}(1-\Gamma_*)\,\tilde{\epsilon}^i = \begin{pmatrix} 0 \\ 0 \\ \epsilon_3^i \\ \epsilon_4^i \\ \epsilon_1^i \\ \epsilon_2^i \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.44}$$

where the tilde denotes the six-dimensional spinors, which are chiral (or antichiral).

[3]We use again $\tilde{\epsilon}$, $\tilde{\zeta}$ to denote six-dimensional quantities, like in the four-dimensional discussion. As we will not directly link 6- and 4-dimensional quantities, we hope that this will not lead to confusion.

The six-dimensional reality conditions are consistent with the five-dimensional ones, due to the choice of the antisymmetric charge conjugation matrix. One can then take the following supersymmetry transformation rules for the six-dimensional hypermultiplet:

$$\begin{aligned}\delta(\tilde{\epsilon})q^X &= -\mathrm{i}\bar{\tilde{\epsilon}}^i\tilde{\zeta}^A f^X_{iA},\\ \delta(\tilde{\epsilon})\tilde{\zeta}^A &= \tfrac{1}{2}\mathrm{i}\Gamma^a\partial_a q^X f^{iA}_X\tilde{\epsilon}^j\varepsilon_{ji} - \tilde{\zeta}^B\omega_{XB}{}^A(\delta(\tilde{\epsilon})q^X).\end{aligned} \tag{2.45}$$

One immediately sees that one can take the vielbeins to be the same functions of the scalars as in $d = 5$. The identification between the vielbeins and spin connections in the 5- and 6-dimensional formalism is thus very simple.

2.4.2 Geometrical aspects

The fact that the identifications between vielbeins and spin connections are trivial shows that the geometrical relations of the 5-dimensional formulation should also hold in the 6-dimensional formulation. Indeed, one can calculate the commutator of 2 supersymmetry transformations on the scalars and make sure that it gives a translation and closes. The conclusions are the same as in the 5-dimensional case. The vielbeins obey the constraints (2.5). As in the five-dimensional case, these are the only restrictions on the target space of the scalars, coming from imposing the supersymmetry algebra. One can take the same definition (2.6) for the complex structures. Integrability conditions on the vielbeins and cyclicity properties of the Riemann tensor lead to the same curvature relations (2.10) and conclusions about the tensor W (2.11). Again one obtains a hypercomplex manifold.

2.4.3 Dynamical aspects

As already mentioned, the commutator of 2 supersymmetries on the fermions does not lead to new geometrical restrictions. Non-closure terms will lead to equations of motion in the geometry. The calculation is simpler than in the 5-dimensional case, due to the chirality of the spinors. We get

$$[\delta(\tilde{\epsilon}_1),\delta(\tilde{\epsilon}_2)]\tilde{\zeta}^A = \tfrac{1}{2}\bar{\tilde{\epsilon}}_2\Gamma^a\tilde{\epsilon}_1\partial_a\tilde{\zeta}^A - \tfrac{1}{4}\Gamma_a\tilde{\Gamma}^A\bar{\tilde{\epsilon}}_2\Gamma^a\tilde{\epsilon}_1. \tag{2.46}$$

The function $\tilde{\Gamma}^A$ we introduced is given by

$$\tilde{\Gamma}^A = \Gamma^a\mathcal{D}_a\tilde{\zeta}^A + \tfrac{1}{2}W_{CDB}{}^A\tilde{\zeta}^B\bar{\tilde{\zeta}}^D\tilde{\zeta}^C. \tag{2.47}$$

This function $\tilde{\Gamma}^A$ prevents the algebra from closing. Therefore we again obtain the equations of motion $\tilde{\Gamma}^A = 0$.

3 From vector multiplet to hypermultiplet

As explained in the introduction, we will now consider how rigid Kähler manifolds in 4-dimensional theories give rise to hyper-Kähler manifolds upon dimensional reduction.

We start from the four-dimensional vector multiplet. The geometry associated with the scalars in this model is known as special geometry. Rigid special geometry was first introduced in [18,19]. Its ingredients are summarized in subsection 3.1. In subsection 3.2 the dimensional reduction of this model to 3 dimensions, giving rise to a hypermultiplet [16] with the associated hyper-Kähler

geometry, is reviewed. In subsection 3.3 we obtain this geometry in the language that we used before in 5 (or in 6) dimensions. This is the most symmetric way to describe hypercomplex geometry. We can thus obtain curvature tensors and W-tensors for special hyper-Kähler manifolds. The geometry of the vector multiplet is characterized by a holomorphic function $F(X)$. In this way, we calculate curvature tensors of special hyper-Kähler manifolds in terms of this function.

3.1 The $N = 2$, $d = 4$ vector multiplet

The four-dimensional vector multiplet consists of a complex scalar X, a vector A_μ and a fermion field Ω_i. The fermion field carries a chiral SU(2) index $i = 1, 2$. We take the convention that spinors with a lower index have positive chirality, while spinors with an upper index have negative chirality. The supersymmetric Lagrangian for n vector multiplets can be written in terms of a holomorphic function $F(X)$. The arguments X^I $(I = 1, \cdots, n)$ refer to the complex scalar fields of the n vector multiplets. We restrict our attention to Abelian vector multiplets, as the non-Abelian part does not modify the geometry. The Lagrangian is a trivial truncation from that written for superconformal tensor calculus in [13]:

$$\begin{aligned}\mathcal{L}_F \;=\;& \mathrm{i}\partial_\mu F_I \partial^\mu \bar{X}^I + \tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J\,\mu\nu} + \mathrm{i}F_{IJ}\bar{\Omega}^I_i \not{D}\Omega^{iJ} \\ & -\tfrac{1}{8}\mathrm{i}F_{IJ}Y^I_{ij}Y^{ij\,J} + \tfrac{1}{4}\mathrm{i}F_{IJK}Y^{ij\,I}\bar{\Omega}^J_i\Omega^K_j \\ & -\tfrac{1}{8}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}^I_i\gamma^{\mu\nu\rho}\mathcal{F}^{-J}_{\mu\nu\rho}\Omega^K_j + \tfrac{1}{12}\mathrm{i}F_{IJKL}\varepsilon^{ij}\varepsilon^{k\ell}\bar{\Omega}^I_i\Omega^J_\ell\bar{\Omega}^K_j\Omega^L_k + \text{h.c.}\end{aligned} \tag{3.1}$$

$F_{I_1\cdots I_k}$ denotes the kth derivative of F. The Hermitian conjugate is taken by complex conjugation on the bosonic quantities, raising or lowering the i, j indices on the spinor Ω and replacing anti-self-dual with self-dual tensors

$$\mathcal{F}^{\pm I}_{\mu\nu} = \tfrac{1}{2}\left(\mathcal{F}^I_{\mu\nu} \mp \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma\,I}\right), \qquad \mathcal{F}^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]}. \tag{3.2}$$

The complex scalars X^I parametrize an n-dimensional target space with metric

$$g_{I\bar{J}} = N_{IJ} \equiv -\mathrm{i}F_{IJ} + \mathrm{i}\bar{F}_{IJ}, \qquad N^{IJ} \equiv [N^{-1}]^{IJ}. \tag{3.3}$$

This is a Kähler space as one can derive the metric from a Kähler potential

$$g_{I\bar{J}} = \frac{\partial^2 K(X,\bar{X})}{\partial X^I \partial \bar{X}^J}, \qquad K(X,\bar{X}) = \mathrm{i}X^I\bar{F}_I(\bar{X}) - \mathrm{i}\bar{X}^I F_I(X). \tag{3.4}$$

This geometry is known as *rigid special Kähler geometry.*

The supersymmetry transformations for the vector multiplet are

$$\begin{aligned}\delta X^I \;&=\; \bar{\tilde{\epsilon}}^i\Omega^I_i, \\ \delta A^I_\mu \;&=\; \varepsilon^{ij}\bar{\tilde{\epsilon}}_i\gamma_\mu\Omega^I_j + \varepsilon_{ij}\bar{\tilde{\epsilon}}^i\gamma_\mu\Omega^{jI}, \\ \delta\Omega^I_i + \Gamma^I_{JK}\delta X^J\Omega^K_i \;&=\; 2\not{\partial}X^I\tilde{\epsilon}_i - \tfrac{1}{2}\mathrm{i}\varepsilon_{ij}\gamma^{\mu\nu}\tilde{\epsilon}^j N^{IJ}\mathcal{G}^-_{\mu\nu J} + \tfrac{1}{2}\mathrm{i}N^{IJ}\bar{F}_{JKL}\bar{\Omega}^{kK}\Omega^{\ell L}\varepsilon_{ik}\varepsilon_{j\ell}\tilde{\epsilon}^j,\end{aligned} \tag{3.5}$$

with $\mathcal{G}^-_{\mu\nu I}$ an anti-self-dual tensor defined as

$$\mathcal{G}^-_{\mu\nu I} = \mathrm{i}N_{IJ}\mathcal{F}^{-J}_{\mu\nu} - \tfrac{1}{4}F_{IJK}\bar{\Omega}^J_i\sigma_{\mu\nu}\Omega^K_j\varepsilon^{ij}. \tag{3.6}$$

As in section 2.2, supersymmetry parameters $\tilde{\epsilon}^i$ and $\tilde{\epsilon}_i$ have respectively positive and negative chirality.

3.2 Reduction to $d = 3$

In this section, we pursue by reviewing the reduction to 3 spacetime dimensions. A complete discussion can be found in [16]. In reducing, the complex scalar X^I loses its dependence from the fourth spacetime coordinate. An extra scalar A^I from the $\mu = 3$ component of the vector appears in the model. One also obtains a three-dimensional vector. However, in 3 dimensions vectors are dual to scalars. This scalar can be brought into the model by use of a Lagrange multiplier. One adds a Lagrange multiplier term $B_I\varepsilon^{\mu\nu\rho}\partial_\mu F^I_{\nu\rho}$ to the Lagrangian, in order to impose the Bianchi identity. Integrating the field strengths, one has introduced the extra scalar B_I in the Lagrangian. In the case of n vector multiplets, we thus obtain $4n$ real scalars in the reduction process. The resulting geometry is that of a hyper-Kähler manifold, i.e. the restriction of the hypercomplex manifolds considered in section 2 to manifolds with a suitable metric. As we work here with a Lagrangian, its kinetic terms of the scalars guarantee that such a metric exists.

Reducing the spinors gets somewhat more complicated. In four dimensions the spinors are Majorana spinors with four components. In the reduction process, each spinor will split up into 2 spinors with 2 components. However we would like to keep on working with spinors with four components. We will take the following representation of the 3-dimensional Clifford algebra

$$\gamma^\mu = \gamma^\mu_{(4)}\tilde{\gamma}, \qquad (\mu = 0, 1, 2), \tag{3.7}$$

in which $\gamma^\mu_{(4)}$ are the four-dimensional γ-matrices. The matrices γ^μ are an alternative set of 4×4 matrices satisfying the three-dimensional Clifford algebra and commuting with the remaining γ^3. The matrix $\tilde{\gamma}$ contains the unused four-dimensional γ-matrices and is defined as

$$\tilde{\gamma} \equiv -\mathrm{i}\gamma^3\gamma^5, \tag{3.8}$$

where $\gamma^3 = \gamma^3_{(4)}, \gamma^5 = \gamma^5_{(4)}$. The three-dimensional charge conjugation matrix $\mathcal{C}_3$ is given by

$$\mathcal{C}_3 = \mathcal{C}_4\tilde{\gamma}. \tag{3.9}$$

The reduced Lagrangian can be found in [16]. Keeping only the kinetic terms of the scalars we obtain

$$\begin{aligned} \mathcal{L} &= \mathrm{i}(\partial_\mu F_I \partial^\mu \bar{X}^I - \partial_\mu \bar{F}^I \partial^\mu X^I) \\ &\quad - N^{IJ}(\partial_\mu B_I - F_{IK}\partial_\mu A^K)(\partial^\mu B_J - \bar{F}_{JM}\partial^\mu A^M) + \cdots . \end{aligned} \tag{3.10}$$

One can also reduce the supersymmetry transformations. This leads to

$$\begin{aligned} \delta X^I &= -\mathrm{i}\bar{\tilde{\epsilon}}^i\gamma_3\Omega^I_i, \\ \delta A^I &= \mathrm{i}\varepsilon^{ij}\bar{\tilde{\epsilon}}_i\Omega^I_j - \mathrm{i}\varepsilon_{ij}\bar{\tilde{\epsilon}}^i\Omega^{jI}, \\ \delta B_I &= \mathrm{i}F_{IJ}\varepsilon^{ij}\bar{\tilde{\epsilon}}_i\Omega^J_j - \mathrm{i}\bar{F}_{IJ}\varepsilon_{ij}\bar{\tilde{\epsilon}}^i\Omega^{jJ}, \\ \delta\Omega^I_i &= 2\mathrm{i}\gamma^\mu\partial_\mu X^I\gamma_3\tilde{\epsilon}_i + 2N^{IJ}(\gamma^\mu\partial_\mu B_J - \bar{F}_{JK}\gamma^\mu\partial_\mu A^K)\varepsilon_{ij}\tilde{\epsilon}^j \\ &\quad + \mathrm{i}N^{IJ}\delta F_{JK}\Omega^K_i - N^{IJ}\bar{F}_{JKL}N^{KM}(\delta B_M - F_{MN}\delta A^N)\varepsilon_{ij}\gamma_3\Omega^{Lj}, \\ \delta\Omega^{Ii} &= -2\mathrm{i}\gamma^\mu\partial_\mu\bar{X}^I\gamma_3\tilde{\epsilon}^i + 2N^{IJ}(\gamma^\mu\partial_\mu B_J - F_{JK}\gamma^\mu\partial_\mu A^K)\varepsilon^{ij}\tilde{\epsilon}_j \\ &\quad - \mathrm{i}N^{IJ}\delta\bar{F}_{JK}\Omega^{Ki} - N^{IJ}F_{JKL}N^{KM}(\delta B_M - \bar{F}_{MN}\delta A^N)\varepsilon^{ij}\gamma_3\Omega^L_j. \end{aligned} \tag{3.11}$$

Note that we keep the fermion fields in their original four-dimensional form. They are doublets of $\frac{1}{2}(1 \pm \gamma_5)$ projections of four-dimensional Majorana spinors. However, the definition of the conjugate of a spinor has been modified according to the explained rules.

3.3 Back to $d = 5$

In this section, we will lift the supersymmetry transformation rules for the 3-dimensional hypermultiplet up to 5 dimensions. In this way, we are able to write down the corresponding geometrical quantities, associated with the hyper-Kähler geometry, such as the vielbeins and the spin connections in this specific case.

We use the following identification for the spinors in order to find transformation rules that are consistent with those of the five-dimensional hypermultiplet:

$$\zeta^{1I} = \gamma_3 \Omega^{1I} + \Omega^{2I}, \qquad \zeta^{2I} = \Omega_2^I - \gamma_3 \Omega_1^I, \tag{3.12}$$

where we used ζ to denote the five-dimensional spinors. For the supersymmetry parameters, we can give an analogous identification

$$\epsilon^1 = \gamma_3 \tilde{\epsilon}_1 + \tilde{\epsilon}_2, \qquad \epsilon^2 = \tilde{\epsilon}^2 - \gamma_3 \tilde{\epsilon}^1, \tag{3.13}$$

where we used ϵ to denote the five-dimensional parameters. Using the fact that $\mathcal{C}_3 = \mathcal{C}_4 \tilde{\gamma} = \mathrm{i}\mathcal{C}_5 \gamma_3$, we can obtain the following transformation rules for the scalars:

$$\begin{aligned}
\delta x^I &= \frac{1}{2}\bar{\epsilon}^2 \zeta^{2I} - \frac{1}{2}\bar{\epsilon}^1 \zeta^{1I}, \\
\delta y^I &= -\frac{\mathrm{i}}{2}\bar{\epsilon}^2 \zeta^{2I} - \frac{\mathrm{i}}{2}\bar{\epsilon}^1 \zeta^{1I}, \\
\delta A^I &= -\bar{\epsilon}^1 \zeta^{2I} - \bar{\epsilon}^2 \zeta^{1I}, \\
\delta B_I &= -F_{IJ}\bar{\epsilon}^1 \zeta^{2J} - \bar{F}_{IJ}\bar{\epsilon}^2 \zeta^{1J},
\end{aligned} \tag{3.14}$$

where $x^I = \mathrm{Re}\, X^I$ and $y^I = \mathrm{Im}\, X^I$.

3.4 Some geometrical quantities

Now that we have reduced the supersymmetry transformations to five dimensions, we can compare these rules with the general transformation rules for a five-dimensional hypermultiplet. In this way, we can extract the necessary quantities that characterize the hyper-Kähler geometry. We are especially interested in determining the vielbeins and spin connections.

We first compare the transformations (3.14) with the general transformation rules (2.3). Remember that the general index A is here represented by a combination iI, with $i = 1, 2$ and $I = 1, \ldots n$. The vielbeins are

$$\begin{aligned}
f^{11I} &= \mathrm{i}\mathrm{d}x^I + \mathrm{d}y^I \\
f^{12I} &= N^{IJ}\left(\mathrm{d}B_J - \bar{F}_{JK}\mathrm{d}A^K\right) \\
f^{21I} &= N^{IJ}\left(-\mathrm{d}B_J + F_{JK}\mathrm{d}A^K\right) \\
f^{22I} &= -\mathrm{i}\mathrm{d}x^I + \mathrm{d}y^I.
\end{aligned} \tag{3.15}$$

We present the inverse vierbeins, which take a simpler form as a matrix whose rows represent the components

$$\begin{pmatrix} x^I \\ y^I \\ A^I \\ B_I \end{pmatrix}, \tag{3.16}$$

and the columns the (iA) values $(1\,1J)$, $(1\,2J)$, $(2\,1J)$ and $(2\,2J)$:

$$f^X{}_{iA} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\delta^I_J & 0 & 0 & \frac{\mathrm{i}}{2}\delta^I_J \\ \frac{1}{2}\delta^I_J & 0 & 0 & \frac{1}{2}\delta^I_J \\ 0 & -\mathrm{i}\delta^I_J & -\mathrm{i}\delta^I_J & 0 \\ 0 & -\mathrm{i}F_{IJ} & -\mathrm{i}\bar{F}_{IJ} & 0 \end{pmatrix}. \tag{3.17}$$

We can also obtain the spin connection as matrix in the basis $A = (1I), (2I)$ (for the rows), $B = (1J), (2J)$ (for the columns)

$$\omega_{XA}{}^B \mathrm{d}q^X = \begin{pmatrix} N^{JK}\bar{F}_{IKM}\left(\mathrm{i}\mathrm{d}x^M + \mathrm{d}y^M\right) & N^{JK}\bar{F}_{IKL}N^{LM}\left(-\mathrm{d}B_M + F_{MN}\mathrm{d}A^N\right) \\ N^{JK}F_{IKL}N^{LM}\left(\mathrm{d}B_M - \bar{F}_{MN}\mathrm{d}A^N\right) & N^{JK}F_{IKM}\left(-\mathrm{i}\mathrm{d}x^M + \mathrm{d}y^M\right) \end{pmatrix}. \tag{3.18}$$

We now have all the necessary information. The metric can be read off from the Lagrangian (3.10)

$$\begin{aligned} g_{XY}\mathrm{d}q^X\mathrm{d}q^Y &= N_{IJ}\mathrm{d}x^I\mathrm{d}x^J + N_{IJ}\mathrm{d}y^I\mathrm{d}y^J + \tfrac{1}{4}N_{IJ}\mathrm{d}A^I\mathrm{d}A^J \\ &\quad + N^{IJ}\left(\mathrm{d}B_I - \mathrm{Re}\,F_{IK}\mathrm{d}A^K\right)\left(\mathrm{d}B_J - \mathrm{Re}\,F_{JL}\mathrm{d}A^L\right). \end{aligned} \tag{3.19}$$

Using this metric, one can calculate the Levi-Civita connection. Since we are dealing with a hyper-Kähler manifold, this connection coincides with the Obata connection (i.e. the connection defined on hypercomplex manifolds that leaves the complex structures invariant). The complex structures can also be calculated. In a basis where the rows and the columns are represented by (3.16), they are:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & -N_{IJ} \\ 0 & 0 & -2\delta_I^J & -2\,\mathrm{Re}\,F_{IJ} \\ -N^{JL}\,\mathrm{Re}\,F_{LI} & \frac{1}{2}\delta_I^J & 0 & 0 \\ N^{IJ} & 0 & 0 & 0 \end{pmatrix}, \tag{3.20}$$

$$J^2 = \begin{pmatrix} 0 & 0 & -2\delta_I^J & -2\,\mathrm{Re}\,F_{IJ} \\ 0 & 0 & 0 & N_{IJ} \\ \frac{1}{2}\delta_I^J & N^{JL}\,\mathrm{Re}\,F_{LI} & 0 & 0 \\ 0 & -N^{IJ} & 0 & 0 \end{pmatrix}, \tag{3.21}$$

$$J^3 = \begin{pmatrix} 0 & \delta_I^J & 0 & 0 \\ -\delta_I^J & 0 & 0 & 0 \\ 0 & 0 & 2N^{JL}\,\mathrm{Re}\,F_{LI} & 2\,\mathrm{Re}(F_{IL}N^{LK}\bar{F}_{KJ}) \\ 0 & 0 & -2N^{IJ} & -2N^{IK}\,\mathrm{Re}\,F_{JK} \end{pmatrix}. \tag{3.22}$$

Since we are working with a hyper-Kähler manifold, we can also define a fermion metric

$$C_{CD} = \tfrac{1}{2}\varepsilon^{ij} f^X_{iC} g_{XY} f^Y_{jD}, \tag{3.23}$$

which results here in

$$C_{iI\,jJ} = \varepsilon_{ij}N_{IJ}. \tag{3.24}$$

3.5 The W-tensor

The tensor W_{ABCD} with the last index lowered using the metric (3.24) is symmetric. It can be calculated using the curvature relations (2.11). The result can be represented using arbitrary symplectic vectors $A^A = A^{iI}$:

$$\begin{aligned} W_{ABCD}A^A A^B A^C A^D &= -3V_I N^{IJ} V_J - \mathrm{i}A^{1I}A^{1J}A^{1K}A^{1L}\bar{F}_{IJKL} + \mathrm{i}A^{2I}A^{2J}A^{2K}A^{2L}F_{IJKL} \\ V_I &\equiv \bar{F}_{IKL}A^{1K}A^{1L} + F_{IKL}A^{2K}A^{2L} \end{aligned} \tag{3.25}$$

Using relations (2.10) and (2.11), it is clear that the W-tensor determines the other curvature tensors using the vielbeins that were obtained in the previous section.

Therefore, this formula contains the full description of the geometry of the special hyper-Kähler manifolds. Moreover this tensor also contains the information about the dynamical properties of the system, since it appears explicitly in the equations of motion for the fermions (see (2.12)). The equations of motion for the scalars can be derived from those for the fermions by application of a supersymmetry transformation.

4 Conclusions

In the first part of this paper (section 2), we made the connection between formulations of hypermultiplets in 4, 5 and 6 dimensions. We showed how the same geometrical relations appear in different settings and that these are equivalent by giving explicit translation formulae. Thus, this connects different formulations of the geometry of hypercomplex manifolds, appropriate for supersymmetry in different dimensions. We did not assume the existence of an action. This possibility was already shown in the five-dimensional case in [2] through closure of the supersymmetry algebra, and is hereby extended to 4 and 6 dimensions. In every case, we were able to link the vielbeins, appearing in the supersymmetry transformation rules, with the five-dimensional ones. In each dimension we have also obtained equations of motion through closure of the supersymmetry algebra.

In the second part of this paper (section 3), we have further investigated the dimensional reduction of the rigid four-dimensional vector multiplet to 3 dimensions as performed in [16]. Connecting also the 3-dimensional to the 5-dimensional formulation, simplifies identifications of geometric quantities. As the special geometry of 4 dimensions is fixed by the holomorphic function $F(X)$ we obtain the curvatures and invariant tensors of a subclass of hyper-Kähler manifolds also in terms of this function. This subclass are the 'special hyper-Kähler' manifolds. We obtain all results in a uniform language such that the general analysis of properties of hypercomplex manifolds that was made in appendix B of [2] can be applied. In particular we obtain the W-tensor of special hyper-Kähler manifolds.

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